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Two-dimensional cnoidal waves in Kerr-type saturable nonlinear media

Yaroslav V. Kartashov,[1,2] Victor A. Vysloukh,[3] and Lluis Torner[1]
[1]*ICFO–Institute de Ciencies Fotoniques and Department of Signal Theory and Communications, Universitat Politecnica de Catalunya, 08034 Barcelona, Spain*
[2]*Physics Department, M. V. Lomonosov Moscow State University, 119899 Moscow, Russia*
[3]*Departamento de Fisica y Matematicas, Universidad de las Americas, Puebla, Santa Catarina Martir, 72820 Puebla, Cholula, Mexico*
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We report theoretically the existence, classification, and basic properties of families of stationary two-dimensional cnoidal-type waves in bulk Kerr-type saturable nonlinear media. The families of two-dimensional cnoidal-type wave solutions are shown to exhibit richer features than their known one-dimensional counterparts. At low- and high-energy flows, the cnoidal patterns are predicted to be robust enough to be observable experimentally.

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Self-trapping of two-dimensional beams in saturable Kerr-type nonlinear media has been studied extensively during the past years. On physical grounds, several mechanisms are known to lead to saturation, including temperature-dependent reorientation of anisotropic molecules in gases and liquids [1], Lorentz local field corrections [2], population changes in the case of an off-resonant interaction with an atomic system [3], full ionization of laser-produced plasma [4], and some specific cases of photorefractive nonlinearity [5]. Bright ground-state soliton solutions of bulk models with Kerr-type saturable nonlinear media have been shown to be theoretically stable [6] in contrast to pure Kerr media [7], and under appropriate conditions they approximately describe solitons experimentally observed in photorefractive crystals [5]. Simple dark soliton beams do also exist in models with saturable nonlinearity, and are dynamically stable [8]. Higher-order bound states, characterized by different number of nodes, do also exist [9] but they suffer from azimuthal instabilities [6] leading to their decay into sets of ground-state solitons.

One important current line of research in the area of nonlinear waves is the elucidation of complex self-trapped structures. Examples of such structures are soliton clusters [10], which might be termed *soliton molecules*. Following this line, a fascinating question is the existence of extended, periodic, higher-dimensional, self-trapped structures, which might be termed *light crystals*. Besides its fundamental interest, such a concept may be fruitful in such fields as formation of periodic matrices of ultracold atoms or trapped Bose-Einstein condensates [11], Bloch waves in solid-state physics, or the creation of light-induced reconfigurable waveguide arrays in photorefractive crystals [12]. Such self-trapped periodic patterns might be constructed by the nontrivial generalization of one-dimensional cnoidal waves [13]. Related advances in this direction are the recent observation of two-dimensional (2D) soliton-type arrays due to the optically seeded transverse modulation instability of interference fringes in photorefractive crystals [14], the clustering of photorefractive solitons in weakly correlated wave fronts [15], and the formation of two-dimensional discrete solitons in photonic lattices [16]. The crucial feature of the cnoidal wave concept, in contrast to the arrays built of individual solitons, is that cnoidal waves are global periodic solutions of the wave equation, which exist for different values of the contrast, or localization degree of the individual light spots.

In this paper, we study the model of Kerr-type saturable nonlinear media, and find, for the first time, to the best of our knowledge, the theoretical existence and basic properties of families of stationary two-dimensional periodic solutions or 2D cnoidal-type waves. While from a rigorous mathematical point of view such waves are unstable, we show that in the limits of low and high powers, the 2D cnoidal-type wave patterns are robust enough to be observable experimentally.

Propagation of nonlinear light waves in a bulk saturable media is described by the nonlinear Schrödinger equation

$$i\frac{\partial q}{\partial \xi}=-\frac{1}{2}\left(\frac{\partial^2 q}{\partial \eta^2}+\frac{\partial^2 q}{\partial \zeta^2}\right)+\sigma\frac{q|q|^2}{1+S|q|^2}. \qquad (1)$$

Here, $q(\eta,\zeta,\xi)$ is the dimensionless slowly varying amplitude of the light field; transverse η, ζ and longitudinal ξ coordinates are scaled in terms of the spatial period (in such a way that it equals to 2π) and the diffraction length, respectively; S is the saturation parameter; $\sigma=-1(+1)$ for focusing (defocusing) media. Note that if $q(\eta,\zeta,\xi,S)$ is the solution of Eq. (1), then $\chi q(\chi\eta,\chi\zeta,\chi^2\xi,\chi^{-2}S)$ (here χ is the arbitrary scaling factor) is also a solution of this equation. These scaling transformations can be used for searching of cnoidal wave arrays for different values of saturation parameter and period. Equation (1) admits several conserved quantities including energy flow, linear momentum, and Hamiltonian. In the case of periodic cnoidal-type wave, one can introduce the energy flow per period T as $U=\int_{-T/2}^{T/2}\int_{-T/2}^{T/2}|q|^2 d\eta\, d\zeta$.

Under certain conditions, this model describes light beams self-action in photorefractive crystals [5]. However, we stress that in this paper we are interested in the periodic solutions for any nonlinear wave problem described by Eq. (1). Note also that in the general case, the exact stationary solutions of Eq. (1) are not described by Jacoby elliptic functions, however by analogy we term periodic solution found here as *cnoidal* waves.

We look for steady-state solutions of Eq. (1) in the form $q(\eta,\zeta,\xi)=w(\eta,\zeta)\exp(ib\xi)$, where b is the real propagation

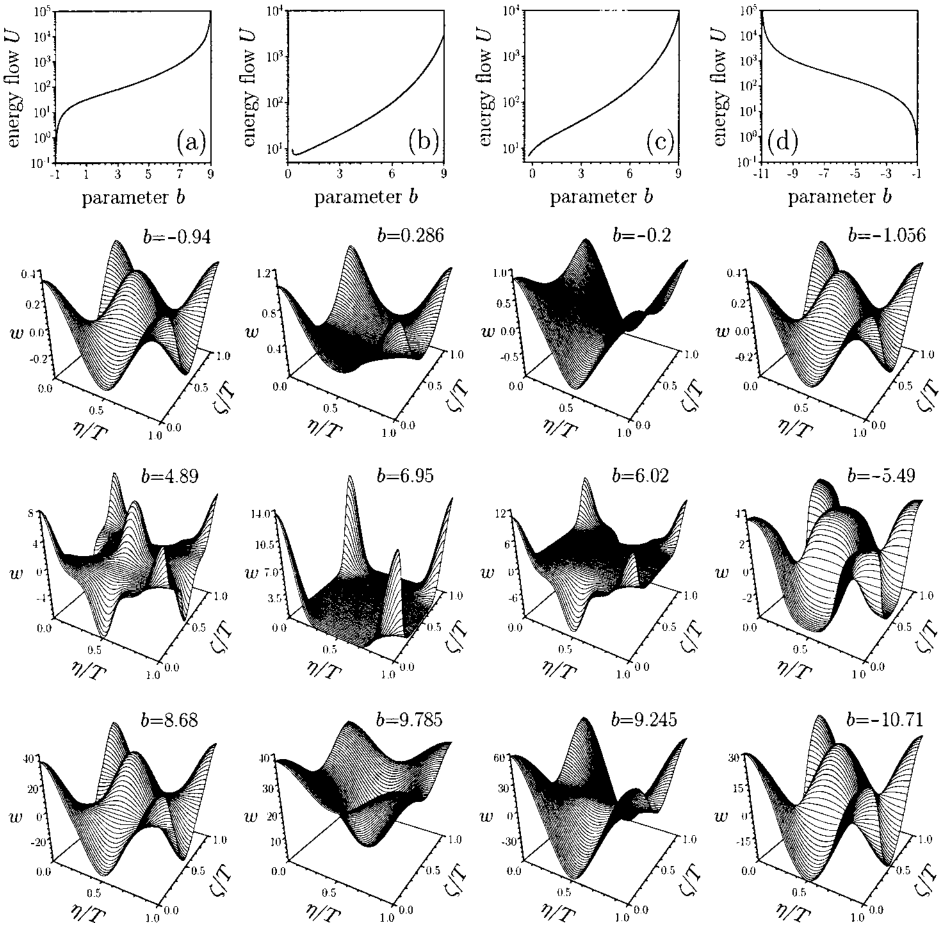

FIG. 1. Cn-cn (a), dn-dn (b), cn-dn (c), and sn-sn (d) cnoidal wave arrays in saturable media. First row shows dispersion diagrams for wave of each type. Surface plots in each column show evolution of wave profile with increase of energy flow U. $T=2\pi$, $S=0.1$.

constant and $w(\eta,\zeta)$ is the real amplitude satisfying periodic boundary conditions $w(\eta+T,\zeta)=w(\eta,\zeta)$, $w(\eta,\zeta+T)=w(\eta,\zeta)$. Substitution of the steady-state wave field into Eq. (1) leads to the following equation for $w(\eta,\zeta)$:

$$\frac{1}{2}\left(\frac{\partial^2 w}{\partial \eta^2}+\frac{\partial^2 w}{\partial \zeta^2}\right)-\frac{\sigma w^3}{1+Sw^2}-bw=0. \qquad (2)$$

Equation (2) admits analytical solutions in two limiting cases of low $w\to 0$ and high $w\to\infty$ field amplitudes. In the first case, the nonlinear term can be neglected and one arrives at the linear Helmholtz-type equation $(1/2)(\partial^2 w/\partial\eta^2+\partial^2 w/\partial\zeta^2)-bw=0$. It has a trivial solution $w(\eta,\zeta)=w_0\cos[(-b)^{1/2}\eta]\cos[(-b)^{1/2}\zeta]$. For high amplitude values $w\to\infty$, one gets the equation $(1/2)(\partial^2 w/\partial\eta^2+\partial^2 w/\partial\zeta^2)-(b+\sigma/S)w=0$ upon linearization of second term in Eq. (3). In this case, the analytical solution has the form $w(\eta,\zeta)=w_0\cos[(-b-\sigma/S)^{1/2}\eta]\cos[(-b-\sigma/S)^{1/2}\zeta]$. These trivial solutions define cutoff values of the propagation constant b, depending on the wave period. For $T=2\pi$, one gets $b_{w\to 0}=-1$ and $b_{w\to\infty}=-1-\sigma/S$.

For arbitrary amplitudes, Eq. (2) has to be solved numerically. For this we used the relaxation technique. It is instructive to classify possible types of solutions using their one-dimensional cross sections. Thus, it is known [13] that for $\partial/\partial\zeta\equiv 0$ and focusing nonlinearity, Eq. (2) has two periodic solutions known as cn and dn waves. The first of these waves periodically changes its sign and has no zero spatial harmonic in its spectrum, whereas the second one is always positive and has zero spatial harmonic in the spectrum. Cn waves were shown to be weakly unstable in cubic medium [13], whereas dn waves are highly unstable due to the presence of zero harmonic in the spatial spectrum. In the defocusing medium, the one-dimensional analog of Eq. (2) admits a solution, called sn wave that is completely stable. Two-dimensional solutions can be divided into four types involving nonlinear combinations of the corresponding one-dimensional waves: cn-cn, dn-dn, and cn-dn waves for focusing nonlinearity, and sn-sn wave for defocusing nonlinearity.

The properties of cn-cn waves are summarized in column (a) of Fig. 1. The energy flow is a monotonically growing function of the propagation constant. At $U\to 0$ and $U\to\infty$, the wave transforms into a two-dimensional harmonic pattern, whereas for intermediate energy levels the wave transforms into an array of out-of-phase solitons. The cutoff values found numerically are in agreement with analytical results presented above.

Dn-dn wave contains zero harmonic in its spatial spectrum [column (b) of Fig. 1]. At $b=0.25$, it transforms into the plane wave $w=[-b/(\sigma+bS)]^{1/2}$. For intermediate energy level, dn-dn wave has a form of array of well-localized in-phase solitons. Near the high-energy cutoff, zero harmonic appears again in the spatial spectrum of dn-dn wave, and field distribution has a rather complicated structure.

Cn-dn wave [column (c) of Fig. 1] at low-energy cutoff

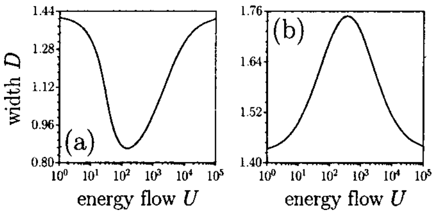

FIG. 2. Width of the separate soliton pixel in the cnoidal wave array versus energy flow for cn-cn (a) and sn-sn (b) arrays. $T=2\pi$, $S=0.1$.

$b=-0.25$ transforms into an array of slit beams, which are uniform along ζ axis and are shaped by cn wave along η axis. At moderate U values, this wave has the form of an array of out-of-phase solitons that differ along ζ axis from the corresponding cn-cn array. Both dn-dn and cn-dn waves have upper limit of energy flow, while energy flow of cn-cn wave is unlimited.

Finally, in the defocusing medium, we have found only one lowest-order cnoidal wave array—sn-sn [column (d) of Fig. 1]. For $U\rightarrow 0$ and $U\rightarrow\infty$, this wave approaches to two-dimensional harmonic pattern, whereas at intermediate energy flows come close to an array of out-of-phase dark solitons.

Among potential applications of the cnoidal waves is the implementation of periodic light spots (e.g., for writing arrays of light-induced waveguides) [16,17]. The integral width of each light spot is defined as

$$D=2\int_0^{T/4}d\eta\int_0^{T/4}d\zeta w^2(\eta,\zeta)(\eta^2+\zeta^2)^{1/2} \Big/ \int_0^{T/4}d\eta\int_0^{T/4}d\zeta w^2(\eta,\zeta). \tag{3}$$

It describes the energy localization within the fixed wave period. The dependence of the integral width on the energy flow is presented in Fig. 2. For the cn-cn array, the width D reaches its minimal value inside the existence segment that corresponds to the highest degree of energy localization. For

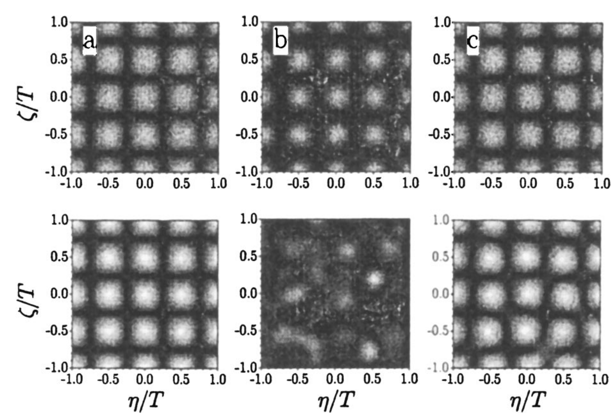

FIG. 3. Propagation of perturbed cn-cn wave with $U=3$ (a), 30 (b), and 30 000 (c). Upper row shows input field distribution, lower row shows field distribution at $\xi=16$. $T=2\pi$, $S=0.1$, noise variance $\sigma_n^2=0.02$.

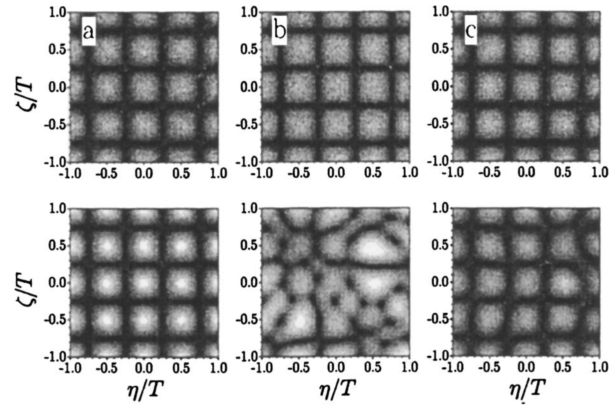

FIG. 4. Propagation of perturbed sn-sn wave with $U=3$ (a), 30 (b), and 30 000 (c). Upper row shows input field distribution, lower row shows field distribution at $\xi=16$. $T=2\pi$, $S=0.1$, noise variance $\sigma_n^2=0.02$.

the sn-sn array, the width reaches its minimal values at low- or high-energy flows, whereas situation where D is maximal corresponds to almost rectangular profile of individual light spot. Note that in the case of self-focusing medium, the minimal possible width of the light spot depends also on the saturation parameter S and decreases as $S\rightarrow 0$. In photorefractive media, the value of the saturation parameter S can be controlled by adjusting the external background illumination.

The cnoidal waves reported here are expected to be unstable from a rigorous mathematical point of view. However, practically the instability growth can be small for actual crystal lengths. Other important question regarding the experimental observation of such arrays is whether they can be excited embedded in laser beams with a finite transverse size. To answer these questions, we solved nonlinear Schrödinger equation (1) with the initial condition $q(\eta,\zeta,\xi=0)=w(\eta,\zeta)F(\eta,\zeta)[1+\rho(\eta,\zeta)]$, where $w(\eta,\zeta)$ describes the profile of stationary cnoidal wave, $F(\eta,\zeta)$ is the broad Gaussian envelope, and $\rho(\eta,\zeta)$ is the Gaussian noise.

Numerical simulations reveal that the two-dimensional cnoidal waves seem to be robust enough to be observed experimentally in the two limiting cases of relatively low- and high-energy flows (see Fig. 3 with examples of propagation of perturbed cn-cn cnoidal wave array). For noise variance $\sigma_n^2=0.01$, the 2D cnoidal wave conserve their input structure for more than 20 diffraction lengths for energy flows lying within rather broad intervals $0<U\leqslant 20$ and $10^2<U<\infty$. The higher the noise level, the faster the instability manifests itself, and the regions of *effective stability* existing at both low and high energies get narrower. For $\sigma_n^2=0.02$, we got approximate intervals $0<U<10$ and $10^3<U<\infty$. For fixed noise level ($\sigma_n^2=0.02$), the cn-cn array with energy flow $U=10^3$ remains almost undistorted up to 20 diffraction lengths (which is of the order of the crystal length), while for $U=10^5$ it survives for more than 30 diffraction lengths and for $U=10^7$ up to almost 50 lengths. Thus, the *effective stability length* increases with nonlinearity saturation.

An analogous behavior was observed in defocusing media for sn-sn-type arrays (Fig. 4) that are unstable in contrast to their $(1+1)$-dimensional counterparts. This result is not surprising in view of the fact that $(2+1)$-dimensional dark soliton stripes are affected by snakelike instabilities [13] in defocusing media.

These observations lead us to the conclusion that under proper conditions of low- and high-energy flows the two-dimensional cnoidal waves appear to be robust enough to be observable in experiments. We would like to mention that one could use for their excitation arrays of Gaussian beams with appropriately adjusted widths and amplitudes, Fourier synthesis of planar waves or holographic techniques. Some of these techniques were already used in photorefractive crystals upon observation of soliton clustering [14,15]. In this context, it must be highlighted that sinusoidal patterns in the photorefractive crystals which might be modeled by low-energy cnoidal-type waves were experimentally observed in Refs. [12], [16] and described in Ref. [17]. Finally, by their very nature the cnoidal waves play an important role in the onset of modulational instabilities in nonlinear systems; thus the existence and properties of the solutions reported here should be instrumental in the full understanding of physical processes mediated by modulational instabilities.

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